Spectral analysis of a self-similar Sturm-Liouville operator

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Abstract: In this text we describe the spectral nature (pure point or continuous) of a self-similar Sturm-Liouville operator on the line or the half-line. This is motivated by the more general problem of understanding the spectrum of Laplace operators on unbounded finitely ramified self-similar sets. In this context, this furnishes the first example of a description of the spectral nature of the operator in the case where the so-called "Neumann-Dirichlet" eigenfunctions are absent.

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In this text we consider a family of self-similar Sturm-Liouville operators, indexed by a parameter ω called the blow-up, and describe the nature of the spectrum (pure point or continuous). This study is motivated by the problem of understanding the nature of the spectrum for the class of self-similar Laplacians on finitely ramified self-similar sets. In [11], [12], [14], we proved that the spectral properties of these operators are related to the dynamics of a certain renormalization map, which is a rational map of a smooth complex projective variety. The main question emerging from these works is the nature of the spectrum of these operators in the non-degenerate case $d_\infty = N$ (cf [12], d_∞ is the asymptotic degree of the renormalization map, N is the number of subcells of the set at level 1). As far as the author knows, there is no non-trivial example where the nature of the spectrum is understood in this case.

The family of self-similar Sturm-Liouville operators we consider here corresponds to the case $d_\infty = N$, and we are able to determine, for all values of the blow-up ω, whether the spectrum is pure point or continuous (we are not able to distinguish the singular continuous and the absolute continuous part). This example also illustrates [13], since it shows that for atypical value of the blow-up ω the spectral nature of the operator can be radically different.

The proofs of the results suggest that the spectral nature of the operators may be related to the way the iterates of the renormalization map approach certain curves (cf remark 4.1). We hope that this example will help to understand the general case of finitely ramified self-similar sets.

The strategy we adopt here to obtain the renormalization equation is a bit different than in [12], and much more straightforward: we write a renormalization equation directly on the propagator of the differential equation associed with the spectral problem. This simplification comes from the 1-dimensional nature of the problem in this example.

1 Definitions and results

Let I be the interval $I = [0,1]$ and α a real such that $0 < \alpha < 1$. We set $\delta = \frac{\alpha}{1-\alpha}$. We define the two homotheties Ψ_1, Ψ_2 by :

$$\Psi_1(x) = \alpha x, \quad \Psi_2(x) = 1 - (1-\alpha)(1-x).$$

So, $\Psi_1(I) = [0,\alpha]$, $\Psi_2(I) = [\alpha, 1]$ and the interval I is self-similar with respect to (Ψ_1, Ψ_2).

Let b be a real number such that $0 < b < 1$. It is classical that there exists a unique probability measure m on I such that

$$\int_0^1 f dm = b \int_0^1 f \circ \Psi_1 dm + (1-b) \int_0^1 f \circ \Psi_2 dm, \quad \forall f \in C(I). \tag{1}$$

N.B.: For $\alpha \neq b$, the measure m is singular with respect to the Lebesgue measure, for $\alpha = b$ it is the Lebesgue measure.

We denote by H^+ the operator $\frac{d}{dm}\frac{d}{dx}$ with Neumann boundary conditions on I, i.e.

it is the operator defined on the domain:

$$\{f \in L^2(I,m),\ \ \exists g \in L^2(I,m), f(x) = ax + b + \int_0^x \int_0^y g(z)dm(z)dy,\\ f'(0) = f'(1) = 0\},$$
$$\text{by} \quad H^+ f = g.$$

We denote by H^- the corresponding operator with Dirichlet boundary conditions on I, i.e. the operator defined on the domain

$$\{f \in L^2(I,m),\ \ \exists g \in L^2(I,m), f(x) = ax + b + \int_0^x \int_0^y g(z)dm(z)dy,\\ f(0) = f(1) = 0\},$$
$$\text{by} \quad H^- f = g.$$

Remark 1.1 : These operators belong to the class of self-similar Laplacians as defined for example in [7] or [10]. Indeed, it is clear that H^+ is the infinitesimal generator associated with the classical Dirichlet form $a(f,g) = \int_0^1 f'g'dx$ defined on $\mathcal{D} = \{f \in L^2(I,m),\ f' \in L^2(I,dx)\}$ and with the measure m. By a change of variables we easily see that a satisfies the following self-similarity relation

$$a(f,g) = \alpha^{-1} a(f \circ \Psi_1, g \circ \Psi_1) + (1-\alpha)^{-1} a(f \circ \Psi_2, g \circ \Psi_2), \quad \forall f,g \in \mathcal{D}.$$

Let $\Omega = \{1,2\}^{\mathbb{N}}$ and let us fix a sequence $\omega = (\omega_1, \ldots, \omega_k, \ldots)$ in Ω. We call this sequence the blow-up. We extend the set I to a set $I_{<n>}(\omega)$ by scaling by

$$I_{<n>}(\omega) = \Psi_{\omega_1}^{-1} \circ \cdots \circ \Psi_{\omega_n}^{-1}(I).$$

(We write $I_{<n>}(\omega)$ to show the dependence of $I_{<n>}$ in ω, but we will simply write $I_{<n>}$ when no ambiguity is possible). Clearly, $I_{<n>}(\omega) \subset I_{<n+1>}(\omega)$ and ω_{n+1} determines the position of $I_{<n>}(\omega)$ in $I_{<n+1>}(\omega)$ (it is the left subinterval of $I_{<n+1>}(\omega)$ when $\omega_{n+1} = 1$ and the right otherwise). Then we set

$$I_{<\infty>}(\omega) = \cup_{n=0}^{\infty} I_{<n>}(\omega).$$

We clearly see that $I_{<\infty>}(\omega)$ is either a half-line bounded from the left if ω is stationary to 1, a half-line bounded from the right if ω is stationary to 2, or the real line if ω is not stationary. We denote by $\partial I = \{0,1\}$ the boundary points of I and by $\partial I_{<n>}(\omega)$, $\partial I_{<\infty>}(\omega)$, the boundary points of $I_{<n>}(\omega)$, $I_{<\infty>}(\omega)$. Of course, $\partial I_{<\infty>}(\omega)$ is empty when ω is not stationary, and contains a unique point when ω is stationary.

We extend the measure m to a measure $m_{<n>}$ on $I_{<n>}(\omega)$ by scaling by

$$\int_{I_{<n>}} f dm_{<n>} = b_{\omega_1}^{-1} \cdots b_{\omega_n}^{-1} \int_I f \circ \Psi_{\omega_1}^{-1} \circ \cdots \circ \Psi_{\omega_n}^{-1} dm, \tag{2}$$

where we set $b_1 = b$, $b_2 = 1-b$. Clearly $b_{<n>}$ is a compatible sequence of measures in the sense that if supp$f \subset I_{<n>}$ then $\int_{I_{<n+p>}} f dm_{<n+p>} = \int_{I_{<n>}} f dm_{<n>}$, for all $p \geq 0$. Hence, it can be extended to a measure $m_{<\infty>}$ on $I_{<\infty>}$.

Then we define $H^+_{<n>}(\omega)$, $H^-_{<n>}(\omega)$ (resp. $H^+_{<\infty>}(\omega)$, $H^-_{<\infty>}(\omega)$) as the operators $\frac{d}{dm_{<n>}} \frac{d}{dx}$ with Neumann or Dirichlet boundary conditions on $I_{<n>}(\omega)$ (resp. $I_{<\infty>}(\omega)$). Of course, when $\partial I_{<\infty>}(\omega) = \emptyset$ then $H^+_{<\infty>}(\omega) = H^-_{<\infty>}(\omega)$ and we simply write $H_{<\infty>}(\omega)$.

From now on, we make the following assumption

(H) We choose $b = 1 - \alpha$, and we set $\gamma = \alpha^{-1}(1-\alpha)^{-1}$.

This choice for b is crucial, it implies that the operator is locally invariant by translation, i.e. that in each subcell of $I_{<n>}$ of level $< p >$, $p \leq n$ (i.e. in each subinterval of the type $\Psi_{\omega_1}^{-1} \circ \cdots \circ \Psi_{\omega_n}^{-1}(\Psi_{j_1} \circ \cdots \circ \Psi_{j_p}(I))$ the restriction of the operator $H_{<n>}$ is the same. This seems to be the natural counterpart of the statistical invariance by translation for random Schrödinger operators (cf [12] for details).

Thanks to the hypothesis (H), we can easily check that the operators $H^{\pm}_{<n>}(\omega)$ are isomorphic for different ω. Indeed, if $\tilde{\Psi}$ is the right composition of Ψ_i and Ψ_i^{-1} that sends $I_{<n>}(\omega)$ to $I_{<n>}(\omega')$, then $H^{\pm}_{<n>}(\omega)(f \circ \tilde{\Psi}) = (H^{\pm}_{<n>}(\omega')(f)) \circ \tilde{\Psi}$. But this is no longer true for $H_{<\infty>}(\omega)$ (in fact, $H_{<\infty>}(\omega)$ and $H^{\pm}_{<\infty>}(\omega')$ are isomorphic if ω and ω' are equal after a certain level, but this is a priori not true otherwise, cf [13] for precisions).

Let us denote by $\nu^{+}_{<n>}$ and $\nu^{-}_{<n>}$ the counting measures of the spectrum of $H^{+}_{<n>}$ and $H^{-}_{<n>}$, respectively. By the previous remark, $\nu^{\pm}_{<n>}$ does not depend on ω. We define the density of states as the limit

$$\mu = \lim_{n \to \infty} \frac{1}{2^n} \nu^{\pm}_{<n>}$$

which exists and does not depend on the boundary condition $\pm$ (cf [3], [8] or [12]).

We denote by $\Sigma^{\pm}(\omega)$ the topological spectrum of $H^{\pm}_{<\infty>}(\omega)$, and by $\Sigma^{\pm}_{ac}(\omega)$, $\Sigma^{\pm}_{sc}(\omega)$, $\Sigma^{\pm}_{pp}(\omega)$ respectively the absolutely continuous, the singular continuous, the purely ponctual part of the Lebesgue decomposition of the spectrum of $H^{\pm}_{<\infty>}(\omega)$. We also denote by $\Sigma^{\pm}_{ess}(\omega)$ the essential spectrum of $H^{\pm}_{<\infty>}(\omega)$. We recall from [13] the following results (proved in the general setting of finitely ramified self-similar sets).

Proposition 1.1 *([13], proposition 1) i) If the boundary set $\partial I_{<\infty>}(\omega)$ is empty, i.e. if ω is not stationary, then $supp\mu = \Sigma(\omega) = \Sigma_{ess}(\omega)$.*

ii) Otherwise we only have $supp\mu = \Sigma^{+}_{ess}(\omega) = \Sigma^{-}_{ess}(\omega)$. Moreover, the eigenvalues eventually lying in $\Sigma^{\pm}(\omega) \setminus supp(\mu)$ have multiplicity 1.

Proposition 1.2 *([13], proposition 2) There exists deterministic sets $\Sigma, \Sigma_{ac}, \Sigma_{sc}$ and Σ_{pp} such that for almost all ω in Ω (for the product of the uniform measure on $\{1,2\}$) we have $\Sigma^{\pm}_{\bullet}(\omega) = \Sigma_{\bullet}$.*

In [11], we gave an explicit expression for the density of states in terms of the Green function of a certain rational map defined on the complex projective plane, and we proved that the spectrum is a Cantor subset of $\mathbb{R}_-$ for $\alpha \neq \frac{1}{2}$. The aim of this text is to go further and to describe the spectral type of the operator $H^{\pm}_{<\infty>}(\omega)$. The main result is the following

Theorem 1.1 *i) If $I_{<\infty>}(\omega)$ is a half-line bounded from the left, i.e. if ω is stationary to 1, then for the Neumann boundary conditions we have*

- *If $\delta > 1$, the spectrum of $H^{+}_{<\infty>}$ is pure point and the eigenvalues lie in the complement of $supp\mu$.*

- *If $\delta < 1$, then $\Sigma^+(\omega) = supp\mu$ and the spectrum of $H^+_{<\infty>}(\omega)$ is continuous, i.e. $\Sigma^+_{pp}(\omega) = \emptyset$.*

For the Dirichlet boundary condition we have

- *If $\delta > 1$, then $\Sigma^-(\omega) = supp\mu$ and the spectrum of $H^-_{<\infty>}(\omega)$ is continuous, i.e. $\Sigma^-_{pp}(\omega) = \emptyset$.*

- *If $\delta < 1$, the spectrum of $H^-_{<\infty>}$ is pure point and the eigenvalues lie in the complement of $supp\mu$.*

ii) If $I_{<\infty>}(\omega)$ is the half-line bounded from the right, i.e. if ω is stationary to 2, then the same results hold, just replacing δ by δ^{-1}.

iii) If the boundary set $\partial I_{<\infty>}(\omega)$ is empty, i.e. if the blow-up ω is not stationary, then $\Sigma(\omega) = supp\mu$ and the spectrum of $H_{<\infty>}(\omega)$ is continuous, that is $\Sigma_{pp}(\omega) = \emptyset$.

2 The renormalization equation of the propagator. Expression of the density of states

We suppose in this section that $\omega = (1, \dots, 1, \dots)$ so that $I_{<\infty>}(\omega) = \mathbb{R}_+$. We denote by $\Gamma_\lambda(s,t)$ the propagator on $[s,t] \subset \mathbb{R}_+$ of the equation

$$\frac{d}{dm_{<\infty>}} \frac{d}{dx} f = \lambda f, \qquad (E_\lambda)$$

for λ in $\mathbb{C}$. This means that if f is a solution of (E_λ) on $[s,t]$ then

$$\begin{pmatrix} f(t) \\ f'(t) \end{pmatrix} = \Gamma_\lambda(s,t) \begin{pmatrix} f(s) \\ f'(s) \end{pmatrix}.$$

Classically, $\Gamma_\lambda(s,t)$ is an element of $Sl_2(\mathbb{C})$ and the coefficients of $\Gamma_\lambda(s,t)$ are analytic in λ. We set $\Gamma_\lambda = \Gamma_\lambda(0,1)$ and $\Gamma_{<n>,\lambda} = \Gamma_\lambda(0, \alpha^{-n})$.

We use the scaling invariance of the operator and the self-similarity to deduce two fondamental relations. Let us introduce the following notation: for a real β we denote by D_β the 2×2 matrix

$$D_\beta = \begin{pmatrix} 1 & 0 \\ 0 & \beta \end{pmatrix}.$$

The scaling relation of m, cf (2), implies

$$\Gamma_{<n>,\lambda} = D_{\alpha^n} \circ \Gamma_{\gamma^n \lambda} \circ D_{\alpha^{-n}}, \qquad (3)$$

where $\gamma = (\alpha(1-\alpha))^{-1}$, cf hypothesis (H). Indeed, if f is a solution of (E_λ) on $I_{<n>}$, then $f \circ \Psi_1^{-n}$ is a solution of $(E_{\gamma^n \lambda})$ on I, and

$$\begin{pmatrix} f(\Psi_1^{-n}(t)) \\ f'(\Psi_1^{-n}(t)) \end{pmatrix} = D_{\alpha^n} \begin{pmatrix} f \circ \Psi_1^{-n}(t) \\ (f \circ \Psi_1^{-n})'(t) \end{pmatrix}$$

for all t in I.
From the self-similarity, and hypothesis (H), we can deduce

$$\begin{aligned}\Gamma_{<1>,\lambda} &= \Gamma_\lambda(1,\alpha^{-1})\circ\Gamma_\lambda(0,1)\\ &= D_\delta\circ\Gamma_\lambda\circ D_{\delta^{-1}}\circ\Gamma_\lambda, \end{aligned}\tag{4}$$

where we recall that $\delta=\frac{\alpha}{1-\alpha}$. Indeed, if f is solution on $I_{<1>}$ of (E_λ), then $\tilde f = f_{|[1,\alpha^{-1}]}\circ\Psi_1^{-1}\circ\Psi_2$ is solution of (E_λ) on $[0,1]$ and

$$\begin{pmatrix}\tilde f(t)\\ \tilde f'(t)\end{pmatrix} = D_{\delta^{-1}}\begin{pmatrix} f(\Psi_1^{-1}\circ\Psi_2(t))\\ f'(\Psi_1^{-1}\circ\Psi_2(t))\end{pmatrix},\quad \forall t\in[0,1].$$

We set

$$\Gamma_{<n>,\lambda}=\begin{pmatrix} a_{<n>}(\lambda) & b_{<n>}(\lambda)\\ c_{<n>}(\lambda) & d_{<n>}(\lambda)\end{pmatrix}$$

and simply $a_{<0>}=a$, $b_{<0>}=b$, $c_{<0>}=c$, $d_{<0>}=d$.

Using (3) and (4) an easy computation gives

$$\begin{aligned}\begin{pmatrix} a_{<1>}(\lambda) & b_{<1>}(\lambda)\\ c_{<1>}(\lambda) & d_{<1>}(\lambda)\end{pmatrix} &= \begin{pmatrix} a(\gamma\lambda) & \alpha^{-1}b(\gamma\lambda)\\ \alpha c(\gamma\lambda) & d(\gamma\lambda)\end{pmatrix}\\ = &\begin{pmatrix} a(\lambda)(a(\lambda)+\delta^{-1}d(\lambda))-\delta^{-1} & b(\lambda)(a(\lambda)+\delta^{-1}d(\lambda))\\ \delta c(\lambda)(a(\lambda)+\delta^{-1}d(\lambda)) & \delta d(\lambda)(a(\lambda)+\delta^{-1}d(\lambda))-\delta\end{pmatrix}\end{aligned}\tag{5}$$

We introduce the polynomial map

$$\begin{aligned} f:\quad \mathbb{C}^2 &\to \mathbb{C}^2\\ (x,y) &\to (x(x+\delta^{-1}y)-\delta^{-1},\delta y(x+\delta^{-1}y)-\delta)\end{aligned}\tag{6}$$

and we see from (5) that the map $\phi:\mathbb{C}\to\mathbb{C}^2$ given by

$$\phi(\lambda)=\begin{pmatrix} a(\lambda)\\ d(\lambda)\end{pmatrix}$$

satisfies

$$f\circ\phi(\lambda)=\phi(\gamma\lambda),\quad \forall\lambda\in\mathbb{C}.\tag{7}$$

Hence, $\{\phi(\lambda),\lambda\in\mathbb{C}\}$ is an invariant complex curve for f.

Proposition 2.1 *(i) The Neumann and Dirichlet spectrum of* $\frac{d}{dm_{<n>}}\frac{d}{dx}$ *are given by*

$$\begin{aligned}\nu^+_{<n>} &= \frac{1}{2\pi}\Delta\ln|c_{<n>}(\lambda)|,\\ \nu^-_{<n>} &= \frac{1}{2\pi}\Delta\ln|b_{<n>}(\lambda)|,\end{aligned}$$

where Δ *denotes the distributional Laplacian on* $\mathbb{C}$.

(ii) The infinite product

$$\prod_{k=1}^{\infty}\alpha(a(\gamma^{-k}\lambda)+\delta^{-1}d(\gamma^{-k}\lambda))$$

is convergent and we have

$$b(\lambda) = \prod_{k=1}^{\infty} \alpha(a(\gamma^{-k}\lambda) + \delta^{-1}d(\gamma^{-k}\lambda)),$$
$$c(\lambda) = \lambda \prod_{k=1}^{\infty} \alpha(a(\gamma^{-k}\lambda) + \delta^{-1}d(\gamma^{-k}\lambda)).$$

Proof: (i) The eigenvalues of $H^+_{<n>}$ and $H^-_{<n>}$ are simple and it is clear that the sets of zeroes of $c_{<n>}$ and $b_{<n>}$ coincide respectively with the sets of eigenvalues of $H^+_{<n>}$ and $H^-_{<n>}$. Thus, the only thing to prove is that the zeroes of $c_{<n>}$ and the zeroes of $b_{<n>}$ are simple. By scaling it is enough to prove it for c and b. Consider a complex number λ such that $\mathrm{Im}\lambda > 0$ and denote by f the solution of (E_λ) on I with initial condition $f(0) = 1$, $f'(0) = 0$. By the Lagrange identity we have

$$[\mathrm{Im}(\overline{f}f')]_0^1 = \mathrm{Im}(\lambda) \int_I |f|^2 dm,$$

which in our case gives

$$\mathrm{Im}(\overline{a}(\lambda)c(\lambda)) = \mathrm{Im}(\lambda) \int_I |f|^2 dm. \tag{8}$$

In particular, this means that $\overline{a}(\lambda)c(\lambda)$ sends the upper-half plane to itself and hence $\overline{a}(\lambda)c(\lambda)$ cannot have a multiple zero on the real axis.

(ii) Remark first that $a(0) = b(0) = d(0) = 1$ and $c(0) = 0$. Hence, there exists K such that $c(\lambda) = K\lambda + O(\lambda^2)$ for λ small. Considering equation (8) for λ small, we deduce that $K = 1$. Indeed, $\overline{a}(0) = 1$ and thus we have $K = \int_I |f|^2 dm$, where f is the solution of $\frac{d}{dm}\frac{d}{dx}f = 0$, with initial condition $f(0) = 1$, $f'(0) = 0$. But this solution is $f = 1$.

From relation (5) we deduce

$$\begin{aligned} b(\lambda) &= b(\gamma^{-n}\lambda) \prod_{k=1}^{n} \alpha(a(\gamma^{-k}\lambda) + \delta^{-1}d(\gamma^{-k}\lambda)) \\ &\underset{n\to\infty}{\sim} \prod_{k=1}^{n} \alpha(a(\gamma^{-k}\lambda) + \delta^{-1}d(\gamma^{-k}\lambda)) \end{aligned}$$

and

$$\begin{aligned} c(\lambda) &= c(\gamma^{-n}\lambda) \prod_{k=1}^{n} (1-\alpha)^{-1}(a(\gamma^{-k}\lambda) + \sqrt{\delta}^{-1}d(\gamma^{-k}\lambda)) \\ &\underset{n\to\infty}{\sim} \lambda \prod_{k=1}^{n} \alpha^{-1}(a(\gamma^{-k}\lambda) + \delta^{-1}d(\gamma^{-k}\lambda)) \end{aligned}$$

This immediately implies that the product is convergent and the formulas for $b(\lambda)$ and $c(\lambda)$.$\diamondsuit$

2.1 The Green function of f. Elements of the dynamics of f.

The map f has a natural extension to the 2-dimensional projective space $\mathbb{P}^2$, given in homogeneous coordinates by

$$f([x, y, z]) = [x(x + \delta^{-1}y) - \delta^{-1}z^2, \delta y(x + \delta^{-1}y) - \delta z^2, z^2]. \tag{9}$$

(The point $[x, y, z]$ represents the image of (x, y, z) in $\mathbb{C}^3$ by the canonical projection $\pi : \mathbb{C}^3 \to \mathbb{P}^2$). Equivalently, this means that f is the map induced on $\mathbb{P}^2$ by the homogeneous polynomial map $R : \mathbb{C}^3 \to \mathbb{C}^3$ given by

$$R((x, y, z)) = (x(x + \delta^{-1}y) - \delta^{-1}z^2, \delta y(x + \delta^{-1}y) - \delta z^2, z^2),$$

the relation being $f(\pi(x)) = \pi(R(x))$. We can see that f is not defined where R is null. We remark that $R((1, -\delta, 0)) = (0, 0, 0)$ and that $\mathbb{C}(1, -\delta, 0)$ is the unique complex line on which R is null. We denote by $l = [1, -\delta, 0]$ the associated point in the projective space and we say that l is a point of indeterminacy. The map f is then a map from $\mathbb{P}^2 \setminus \{l\}$ to $\mathbb{P}^2$ and is holomorphic on $\mathbb{P}^2 \setminus \{l\}$ (in fact the image of the point l by f can be defined as a compact Riemann surface called the blow-up of l, cf ([11]). Therefore, the map f is called a rational map of $\mathbb{P}^2$. Its degree is 2 in relation with the degree of the homogeneous polynomials appearing in R (cf [15]).
We set

$$D = \{[x, y, z], \;\; x + \delta^{-1}y = 0\}. \tag{10}$$

The line D is sent by f to a unique point, $[-\delta^{-1}, -\delta, 1]$, and the orbit of D is

$$f(D \setminus \{l\}) = \{[-\delta^{-1}, -\delta, 1]\}, \;\; \text{and} \;\; f^{n+1}(D \setminus \{l\}) = [\delta^{-2^n}, \delta^{2^n}, 1], \tag{11}$$

for $n \geq 0$. The set D is called a f-constant curve (line), since it is sent to a unique point. It is a general phenomenon that a f-constant curve contains a point of indeterminacy (cf proposition 1.2 of [15]).
Another important property of the map f is that it has no degree lowering curve: a degree lowering curve is a f-constant curve sent by f^n to a point of indeterminacy. When such a phenomenon appears, a common factor, which can be divided out, appears in R^n and the degree of the map f^n drops. Here, we remark that the orbit of D does not contain the indeterminacy point l, so D is not a degree lowering curve. Hence, degree$(f^n) = 2^n$ (this means that R^n is represented by 3 homogeneous polynomials of degree 2^n with no common factor). Following the terminology in [15], f is said to be algebraically stable, cf definition 4.4.
We set $\tilde{I} = \cup_{n \geq 0} f^{-n}(\{l\}) = \{[1, -\delta^{-(n-1)}, 0]\}_{n \geq 0}$ the set of preimages of the point of indeterminacy $\{l\}$.
The Fatou set of f is defined to be the union of all open balls $U \subset \mathbb{P}^2 \setminus \tilde{I}$ on which the family $\{f^n\}_{n \geq 0}$ is normal. The Fatou set is denoted by $\mathcal{F}$ and its complement, the Julia set, by $\mathcal{J} = \mathbb{P}^2 \setminus \mathcal{F}$. Of particular interest to us is the fact that the attractive basin of an attractive fixed point is in the Fatou set.
A function, useful to study the dynamics of f, is the Green function defined as the limit of the sequence of functions $G_n : \mathbb{C}^2 \to \mathbb{R} \cup \{-\infty\}$:

$$G_n(x) = \frac{1}{2^n} \log(1 + \|f^n(x)\|), \quad x \in \mathbb{C}^2. \tag{12}$$

where $\| \;\|$ denotes the usual norm of $\mathbb{C}^2$, and where we considered f as a map on $\mathbb{C}^2$, by (6).
We will use the following result (cf [15], proposition 2.11 or [2], theorem 1.6.1):

Proposition 2.2 *(i) The limit*

$$G(x) = \lim_{n\to\infty} G_n(x)$$

exists for all $x \in \mathbb{C}^2$. *The function* G *is plurisubharmonic and satisfies*

$$G \circ f = 2G. \tag{13}$$

(ii) G *is pluriharmonic on* $\mathbb{C}^2 \cap \mathcal{F}$.

NB: In general for a rational map on $\mathbb{P}^k$ the Green function is defined on $\mathbb{C}^{k+1}$, cf [15]. In our case since f is polynomial it is easier to adopt the previous definition.

Denote by $\zeta(\lambda)$ the Lyapounov exponent of the propagator of the O.D.E. (E_λ) given by

$$\zeta(\lambda) = \lim_{n\to\infty} \frac{1}{2^n} \ln \|\Gamma_\lambda(0, \alpha^{-n})\|$$

when this limit exists. In [11], theorem 3.1 and proposition 3.6, we proved

Theorem 2.1 *(i) The Lyapounov exponent exists and is given by*

$$\zeta(\lambda) = G \circ \phi(\lambda)$$

for all λ *in* $\mathbb{C}$.

(ii) The density of states is given by

$$\mu = \frac{1}{2\pi} \Delta\zeta,$$

where Δ *denotes the distributional Laplacian on* $\mathbb{C}$.

Rm: The point (ii) corresponds to the Thouless formula in our context. In [12] we gave similar formulas in the general case of finitely ramified self-similar sets for the density of states in terms of the Green function of a certain renormalization map.

In [11] we deduced from the analysis of the dynamics of f that $\mathrm{supp}(\mu)$ is a Cantor subset of $\mathbb{R}_-$ for $\delta \neq \frac{1}{2}$, and for certain values of the parameter δ we proved the local Hölder regularity of the Lyapounov exponent.

2.2 Some details about the dynamics of the map f

We give some information about the dynamics of the map f that will be useful in the sequel.

We set $x_+ = [0, 1, 0]$ and $x_- = [1, 0, 0]$. The first important point is that for $\delta > 1$

- x_- have one attractive direction and one repulsive (with eigenvalues 0 and δ).
- x_+ is attractive (with eigenvalues 0 and δ^{-1}).

(and the reverse for $\delta < 1$).

We denote by $\mathcal{C} \subset \mathbb{P}^2$ the hypersurface:

$$\mathcal{C} = \{[x, y, z], \;\; xy = z^2\}. \tag{14}$$

The restriction of $\mathcal{C}$ to $\mathbb{R}^2$ consists of two branches of hyperbolas (cf picture 1). Remark that the restriction of f to $\mathcal{C}$ is given by

$$f([x,y,z]) = [x^2, y^2, z^2], \quad [x,y,z] \in \mathcal{C},$$

and thus

$$f(C) \subset C.$$

We set

$$\mathcal{C}_+ = \{[x,y,z] \in \mathcal{C},\ |x| < |y|\}, \quad \mathcal{C}_- = \{[x,y,z] \in \mathcal{C},\ |x| > |y|\}. \tag{15}$$

Clearly, any point in $\mathcal{C}_\pm$ converges to $x_\pm$. Hence, the set $\mathcal{C}_+$ is in the Fatou set of f for $\delta > 1$ since it is in the attractive basin of x_+. The same is true for the set $D \setminus \{l\}$ since $f^n(D \setminus \{l\})$ converges to x_+ (cf formula (11).

We set

$$K_+ \ \text{(resp. } K_-) = \{[x,y,z],\ (x,y,z) \in \mathbb{R}^3,\ xy - z^2 \geq 0\} \ \text{(resp. } xy - z^2 \leq 0). \tag{16}$$

The following properties are direct consequences of the forthcoming formula 24 (cf [11], formula 3.17)

$$f(K_\pm) \subset K_\pm, \quad \phi(\mathbb{R}_\pm) \subset K_\pm. \tag{17}$$

3 Spectral analysis of the operator on a half-line

In this section, we consider the case where ω is stationary, i.e. where $\partial I_{<\infty>}(\omega) \neq \emptyset$. By the symmetric role played by α and $(1-\alpha)$ it is enough to analyse the case where $I_{<\infty>}(\omega)$ is a half-line bounded from the left, i.e. when ω is stationary to 1. By scaling it is enough to consider the case $\omega = (1, \ldots, 1, \ldots)$. Thus, in this section, we only consider the case $\omega = (1, \ldots, 1, \ldots)$, so that $I_{<\infty>}(\omega) = \mathbb{R}_+$.

Denote by S the set of intersection times of the curve $\phi(\gamma^{-1}\lambda)$ with the line D defined in (10), i.e.

$$S = \{\lambda,\ \ \phi(\gamma^{-1}\lambda) \in D\}.$$

For p in $\mathbb{Z}$ we set

$$S_p = \gamma^p S.$$

We know from proposition 2.1 that the spectrum of $H^+_{<0>}$ and $H^-_{<0>}$ are given by

$$\text{supp}\nu^+_{<0>} \setminus \{0\} = \text{supp}\nu^-_{<0>} = \cup_{p=0}^{\infty} S_p. \tag{18}$$

Hence, we know that S is non-empty and included in $\mathbb{R}_-$. The set S is also infinite. Indeed, if λ_0 is in S then $\phi(\gamma^{-1}\lambda_0)$ is in D and $\phi(\lambda_0) = [-\delta^{-1}, -\delta, 1]$, $\phi(\gamma\lambda_0) = [\delta^{-2}, \delta^2, 1]$, cf (11). Hence, $\phi(\gamma^{-1}\lambda)$ must necessarily cross the line D between $\gamma\lambda_0$ and $\gamma^2\lambda_0$. We denote by $0 > \lambda_1 > \lambda_2 > \ldots > \lambda_k > \ldots$ the ordered set of points of S. We also denote by $\lambda_{k,p} = \gamma^p \lambda_k$ the points of S_p.

For any $k > 0$ we denote by $f_k^\pm$ the solutions of the equation (E_{λ_k}) with initial condition $f_k^+(0) = 1$, $(f_k^+)'(0) = 0$ and $f_k^-(0) = 0$, $(f_k^-)'(0) = 1$. We denote by $f_{k,p}^\pm$, for p in $\mathbb{Z}$, the scaled copy of $f_k^\pm$

$$f_{k,p}^\pm = f_k^\pm \circ \Psi_1^{-p}.$$

By scaling $f_{k,p}^\pm$ is solution of $(E_{\gamma^p\lambda_k})$ on $\mathbb{R}_+$. We now denote by $f_{k,p,<n>}^\pm$ the restriction of $f_{k,p}^\pm$ to $I_{<n>}$.

Lemma 3.1 *(i) The set*

$$\{1,\ f^+_{k,p,<n>},\ k>0,\ p\geq -n\},$$

is a complete set of eigenfunctions of $H^+_{<n>}$.

(ii) The set

$$\{f^-_{k,p,<n>},\ k>0,\ p\geq -n\}$$

is a complete set of eigenfunctions of $H^-_{<n>}$.

Proof: (i) By scaling it is enough to check it for $n=0$. We already know that the eigenvalues of $H^+_{<0>}$ are equal to the set $\{0\}\cup_{p=0}^{\infty} S_p$. The functions $f^+_{k,p,<0>}$ satisfy the Neumann boundary condition in 0 and are solutions of $(E_{\gamma^p\lambda})$ on I, hence $f_{k,p,<0>}$ is necessarily an eigenfunction of $H^+_{<0>}$ on I for $p\geq 0$. Since the eigenvalues of $H^+_{<0>}$ are simple, 1 and $f_{k,p,<0>}$ for $p\geq 0$, $k\geq 1$, do necessarily form a complete family of eigenfunctions of $H^+_{<0>}$. The proof of (ii) is similar.$\diamondsuit$

Theorem 3.1 *(The pure point case)*

(i) If $\delta>1$, *the functions* $\{f^+_{k,p}\}_{k\geq 1, p\in\mathbb{Z}}$ *are in* $L^2(\mathbb{R}_+, m_{<\infty>})$ *and form a complete family of eigenfunctions of* $H^+_{<\infty>}$. *Hence, the spectrum of* $H^+_{<\infty>}$ *is pure point and the set of eigenvalues is*

$$\cup_{p=-\infty}^{\infty} S_p. \tag{19}$$

These eigenvalues lie in the complement of suppμ.

(ii) If $\delta<1$, *the functions* $\{f^-_{k,p}\}_{k>0, p\in\mathbb{Z}}$ *are in* $L^2(\mathbb{R}_+, m_{<\infty>})$ *and form a complete set of eigenfunctions of* $H^-_{<\infty>}$. *Hence the spectrum of* $H^-_{<\infty>}$ *is purely ponctual and the set of eigenvalues is (19).*

Proof: (i) By scaling it is enough to prove that $f^+_{k,p}$ is in L^2 for $p=0$. Considering the orbit of the line D, cf (11), we deduce from (3) and (5) that for a point λ_k in S we have

$$\Gamma_{<0>,\lambda_k} = \begin{pmatrix} -\delta^{-1} & 0 \\ 0 & -\delta \end{pmatrix} \text{ and } \Gamma_{<n>,\lambda_k} = \begin{pmatrix} \delta^{2-n} & 0 \\ 0 & \delta^{2^n} \end{pmatrix}$$

for $n\geq 1$. Thus we have

$$\begin{pmatrix} f^+_k(1) \\ (f^+_k)'(1) \end{pmatrix} = \begin{pmatrix} -\delta^{-1} \\ 0 \end{pmatrix} \text{ and } \begin{pmatrix} f^+_k(\alpha^{-n}) \\ (f^+_k)'(\alpha^{-n}) \end{pmatrix} = \begin{pmatrix} \delta^{-2^n} \\ 0 \end{pmatrix}$$

for $n\geq 1$. Hence for $n\geq 0$ the function $f^+_{k,<n+1>}$ can be written

$$\begin{cases} (f^+_{k,<n+1>})_{|I_{<n>}} = f^+_{k,<n>} \\ (f^+_{k,<n+1>})_{|I_{<n+1>}\setminus I_{<n>}} = b^+_n \left(f^+_{k,<n>} \circ \Psi_1^{-n} \circ \Psi_2^{-1} \circ \Psi_1^{n+1} \right) \end{cases} \tag{20}$$

where $b^+_0 = -\delta^{-1}$ and $b^+_n = \delta^{-2^n}$ for $n\geq 1$. Indeed, the function on the right hand side of the second line is a solution of the equation (E_{λ_k}) on $I_{<n+1>}\setminus I_{<n>}$ and it matches exactly the boundary condition of $f^+_{k,<n>}$. From this we deduce the following

$$\|f^+_{k,<n+1>}\|^2 = \|f^+_{k,<n>}\|^2(1+\delta(b^+_{<n>})^2),$$

where $\| \ \|$ is the L^2 norm with respect to the measure $m_{<n>}$ and $m_{<n+1>}$ (the extra factor δ comes from the scaling relation between $m_{<n>}$ and $(m_{<n+1>})_{|I_{<n+1>}\setminus I_{<n>}}$). For $\delta > 1$, this immediately implies that f_k^+ is in $L^2(m_{<\infty>})$ for all $k > 1$. Remark also that by scaling we deduce the following relation, for all $n \geq 0$ and $p \geq -n$:

$$\|f_{k,p,<n+1>}^+\|^2 = \|f_{k,p,<n>}^+\|^2(1 + \delta(b_{n+p}^+)^2),$$

In particular, we deduce from the previous relation that there exists a constant depending only on δ, $C_\delta = \prod_{k=0}^{\infty}(1 + \delta(b_k^+)^2)$, such that for all $p \geq -n$

$$\|f_{k,p}^+\|^2 \leq C_\delta \|f_{k,p,<n>}\|^2. \tag{21}$$

To prove that the family is complete it is enough to prove that for any function g in $L^2(m_{<\infty>})$ with compact support we have

$$\|g\|^2 = \sum_{k=1}^{\infty} \sum_{p \in \mathbb{Z}} \frac{|\int g f_{k,p}^+ dm_{<\infty>}|^2}{\|f_{k,p}^+\|^2}, \tag{22}$$

and we may as well suppose that supp$g \subset I$ by scaling invariance.
Take a positive ϵ. Since the sum on the right is converging, we can find $n_1 > 0$ such that

$$\sum_{k=1}^{\infty} \sum_{p=-\infty}^{-n_1-1} \frac{|\int g f_{k,p}^+ dm_{<\infty>}|^2}{\|f_{k,p}^+\|^2} \leq \epsilon. \tag{23}$$

Since $\{1, f_{k,p,<n>}^+, k \geq 1, p \geq -n\}$ is a complete family of eigenfunctions for $H_{<n>}^+$ and since supp$g \subset I$, we have:

$$\|g\|^2 = \frac{|\int_{I_{<n>}} g dm_{<n>}|^2}{\int_{I_{<n>}} 1 dm_{<n>}} + \sum_{k=1}^{\infty} \sum_{p=-n}^{\infty} \frac{|\int_{I_{<n>}} g f_{f,p,<n>}^+ dm_{<n>}|^2}{\|f_{k,p,<n>}^+\|^2}.$$

For $n \geq n_1$ we have

$$\begin{aligned} & \left| \|g\|^2 - \sum_{k=1}^{\infty} \sum_{p=-n_1}^{\infty} \frac{|\int_{I_{<\infty>}} g f_{k,p}^+ dm_{<\infty>}|^2}{\|f_{k,p,<n>}^+\|^2} \right| \\ \leq & \ \frac{|\int_I g dm|^2}{\int_{I_{<n>}} 1 dm_{<n>}} + \sum_{k=1}^{\infty} \sum_{p=-n}^{-n_1-1} \frac{|\int_{I_{<\infty>}} g f_{k,p}^+ dm_{<\infty>}|^2}{\|f_{k,p,<n>}^+\|^2} \\ \leq & \ \frac{|\int_I g dm|^2}{(1-\alpha)^{-n}} + \epsilon C_\delta. \end{aligned}$$

In the last equation we used relation (21) and (23). Letting n go to infinity we get

$$\left| \|g\|^2 - \sum_{k=1}^{\infty} \sum_{p=-n_1}^{\infty} \frac{|\int_{I_{<\infty>}} g f_{k,p}^+ dm_{<\infty>}|^2}{\|f_{k,p}^+\|^2} \right| \leq \epsilon C_\delta.$$

Letting ϵ go to zero we prove relation (22).

To prove that $\cup_{p\in\mathbb{Z}}S_p$ is in the complement of suppμ, we simply use the fact that the line $\mathbb{C}^2 \cap D$ is in the attractive bassin of the point $[0,1,0]$ (which is attractive for $\delta > 1$, cf [11]), and hence is in the Fatou set of f. Thus, μ must be null in a neighborhood of $\cup_{p\in\mathbb{Z}}S_p$ since G is pluriharmonic in a neighborhood of $\mathbb{C}^2 \cap D$.

(ii) The proof for the Dirichlet boundary condition is similar. We prove a similar relation on the norm of the functions $f^-_{k,p,<n>}$ involving a sequence of coefficients b^-_n given by $b^-_0 = -1$, $b^-_n = \delta^{2^n-1}$. Then the proof goes exactly the same way.$\diamondsuit$

Theorem 3.2 *(The continuous case)*

(i) If $\delta < 1$, then in the case of the operator with Neumann boundary conditions $\Sigma^+ =$ suppμ, and the spectrum of $H^+_{<\infty>}$ is continuous.

(ii) If $\delta > 1$ then in the case of the operator with Dirichlet boundary condition $\Sigma^- =$ suppμ, and the spectrum of $H^-_{<\infty>}$ is continuous.

Proof: We first prove the following lemma.

Lemma 3.2 *For a stationary blow-up ω, we have*

$$\Sigma^\pm \subset \cup_{p\in\mathbb{Z}}S_p \sqcup \mathit{supp}\mu.$$

Remark 3.1 : This result is actually true for any blow-up ω, but for a non-stationary blow-up ω we proved in proposition 1.1, that $\Sigma(\omega) = \text{supp}\mu$. Comparing this lemma with proposition 1.1, we see that it can be rephrased as $\Sigma^\pm \setminus \Sigma^\pm_{ess} \subset \cup_{p\in\mathbb{Z}}S_p$.

Proof: From general results and formula (18) we know that

$$\begin{aligned} \Sigma^\pm &\subset \cap_{n\in\mathbb{N}}\overline{\cup_{m\geq n}\text{supp}\nu^\pm_{<m>}}, \\ &= \overline{\cup_{p\in\mathbb{Z}}S_p} \end{aligned}$$

Let us first prove that the points of S_p are isolated in $\cup_{p\in\mathbb{Z}}S_p$. By scaling it is enough to prove it for $p = 0$. For $\lambda_k \in S$, $\phi(\gamma^{-1}\lambda_k)$ is in D and thus in the attractive basin of x_+ for $\delta > 1$ (and x_- for $\delta < 1$). We can find a neighborhood U of λ_k such that $f^n(\phi(\gamma^{-1}U)) \cap D = \emptyset$ for all $n > 0$. This implies that $U \cap (\cup_{p<0}S_p) = \emptyset$. To see that λ_k is isolated in $\cup_{p>0}S_p$ it is enough to remark that $\cup_{p>0}S_p$ has no accumulation point since for any $R > 0$ there exists N large enough such that $S_p \cap B(0,R) = \emptyset$ for $p \geq N$ ($B(0,R)$ is the ball in $\mathbb{C}$ with center 0 and radius R). From this we deduce

$$\Sigma^\pm \subset (\cup_{p\in\mathbb{Z}}S_p) \sqcup (\cap_{n\in\mathbb{N}}\overline{\cup_{m\leq -n}S_m}).$$

Take now λ in $\cap_{n\in\mathbb{N}}\overline{\cup_{m\leq -n}S_m}$. Suppose that λ is not in suppμ. This means that there exists a small open ball U around λ such that $\mu(U) = 0$, thus that $G \circ \phi$ is harmonic on U. Mimicking the proof of [15], Theorem 6.5, we see that this implies that the family of functions $(f^n \circ \phi)_{n\in\mathbb{N}}$ is normal on U. But necessarily U contains a point of S_p for a certain $p < 0$, which is in the attractive basin of x_+ for $\delta > 1$ (or x_- for $\delta < 1$). This implies that U itself is included in the attractive basin of x_+ or x_-. Considering a neighborhood V of x_+ (or x_-) such that $V \cap D = \emptyset$, we know that there exists N such that $f^n(U) \subset V$ for $n \geq N$. Thus $U \cap S_p = \emptyset$ for $p \leq -(N+1)$ and this is contradictory.$\diamondsuit$

To prove i) of theorem 3.2 we first prove that for $\delta < 1$, $\lambda_{k,p}$ cannot be an eigenvalue of $H^+_{<\infty>}$. But this is clear from relation (20) since $f^+_{k,p}$ is not in $L^2(m_{<\infty>})$. The only thing which remains to prove is that that λ in suppμ cannot be an eigenvalue of $H^\pm_{<\infty>}$. This will be done in lemma 4.1 in the next section.

The proof of ii) is strictly similar.

4 The continuous spectrum

We prove the following lemma.

Lemma 4.1 *For any blow-up ω, λ in suppμ cannot be an eigenvalue of $H^{\pm}_{<\infty>}(\omega)$.*

As we explained at the end of the previous section, this lemma concludes the proof of Theorem 3.2. Thanks to proposition 1.1, it also concludes the proof of theorem 1.1, iii) (the case of a non-stationary blow-up ω).
We first prove

Lemma 4.2 *For any λ in suppμ, there exists a constant C_1, depending only on λ and δ such that for all n in $\mathbb{N}$*

$$\|f^n(\phi(\lambda))\| \leq C_1.$$

Remark 4.1 : Let us remark that this implies that the iterates of $\phi(\lambda)$ cannot approach the indeterminacy point of f, which is located on the curve at infinity $[x, y, 0]$. It seems intuitively natural that this condition is related to continuous spectrum since indeterminacy points correspond to eigenfunctions with compact support (cf [12]). Hence, eigenfunctions with non-compact support could be related to the way an orbit can approach the set of indeterminacy points.
Proof: If $\lambda = 0$, then $\phi(\lambda)$ is fixed by f. Let us take λ in supp$\mu \setminus \{0\}$. We set $(x_0, y_0) = \phi(\lambda)$. The vector (x_0, y_0) is in K_- (cf section 2.2). The result of lemma 4.2 concerns only the real dynamics of $f : \mathbb{P}^2_{\mathbb{R}} \to \mathbb{P}^2_{\mathbb{R}}$, where $\mathbb{P}^2_{\mathbb{R}}$ is the 2-dimensional real projective space.
We set for any (x, y, z) in $\mathbb{R}^3$, as in [11],

$$r((x, y, z)) = xy - z^2$$
$$p((x, y, z)) = \alpha(x + \delta^{-1}y).$$

For any $\epsilon > 0$, we set

$$V_\epsilon = \{[x, y, z] \in \mathbb{P}^2_{\mathbb{R}}, \ \ |r(\frac{(x, y, z)}{\|(x, y, z)\|})| \leq \epsilon\}.$$

The family $(V_\epsilon)_{\epsilon>0}$ gives a base of neighborhoods of $C \cap \mathbb{P}^2_{\mathbb{R}}$. We first prove the following lemma

Lemma 4.3 *There exists $\epsilon > 0$ such that the iterates $f^n((x_0, y_0))$ do not enter V_ϵ, i.e.*

$$(\cup_{n\geq 0}\{f^n((x_0, y_0))\}) \cap V_\epsilon = \emptyset.$$

Proof: The set C_+ is in the attractive basin of x_+. Let V_+ be a neighborhood in $\mathbb{P}^2_{\mathbb{R}}$ of $(C_+ \cup \{x_+\}) \cap \mathbb{P}^2_{\mathbb{R}}$ contained in the attractive basin of x_+. Since (x_0, y_0) is in the Julia set of f, the iterates $f^n((x_0, y_0))$ do not enter the set V_+. Remark that the closure in $\mathbb{P}^2_{\mathbb{R}}$ of $C_- \cap \mathbb{P}^2_{\mathbb{R}}$ is given by

$$\overline{C_- \cap \mathbb{P}^2_{\mathbb{R}}} = \{x_-\} \cup \{[x, \frac{1}{x}, 1], \ \ x \in \mathbb{R}_*, \ \ |x| \geq 1\}.$$

We now use, as in [11], (3.27), the following formula

$$r \circ R(X) = \frac{1}{\alpha(1-\alpha)}(p(X))^2 r(X), \quad \forall X \in \mathbb{R}^3, \tag{24}$$

which implies

$$\left| r\left(\frac{R(X)}{\|R(X)\|}\right)\right| = \frac{|r \circ R(X)|}{\|R(X)\|^2} = \frac{1}{\alpha(1-\alpha)}\frac{\|X\|^2|p(X)|^2}{\|R(X)\|^2}\left| r\left(\frac{X}{\|X\|}\right)\right|^2, \quad \forall X \in \mathbb{R}^3. \tag{25}$$

Take $X = (x, y, z)$ s.t. $\pi(X) \in \overline{C_- \cap \mathbb{R}^2}$. We have

$$\frac{\|X\|^2}{\|R(X)\|^2}\frac{1}{\alpha(1-\alpha)}|p(X)|^2 \quad = \quad \frac{x^2+y^2+z^2}{x^4+y^4+z^4}\delta(x+\delta^{-1}y)^2.$$

If $\pi(X) = x_-$ then the last value equals δ. Otherwise, we may as well suppose, by homogeneity, that $z = 1$, and then we get, using the fact that $|x| \geq 1$,

$$\begin{aligned}\frac{x^2+\frac{1}{x^2}+1}{x^4+\frac{1}{x^4}+1}\delta(x+\delta^{-1}\frac{1}{x})^2 \quad &\geq \quad \delta\frac{x^2(x^2+\frac{1}{x^2}+1)}{x^4+\frac{1}{x^4}+1} \\ &\geq \quad \delta > 1.\end{aligned}$$

Thus, we can find a neighborhood $\tilde{V}_-$ of $\overline{C_- \cap \mathbb{P}^2_{\mathbb{R}}}$, in $\mathbb{P}^2_{\mathbb{R}}$, such that

$$|r(\frac{R(X)}{\|R(X)\|})| \geq |r(\frac{X}{\|X\|})|, \quad \forall X \in \mathbb{R}^3, \text{ s.t. } \pi(X) \in \tilde{V}_-. \tag{26}$$

Using formula (24) we see that

$$f^{-1}(C) \subset C \cup D.$$

But since $f(D) = [-\delta^{-1}, -\delta, 1] \in C_+$, and since $f([x,y,z]) = [x^2, y^2, z^2]$ for $[x,y,z] \in C$, we see that $f^{-1}(C_-) \subset C_-$. Considering the map f only on $\mathbb{P}^2_{\mathbb{R}}$, we see that we can find a neighborhood V_- of $\overline{C_- \cap \mathbb{P}^2_{\mathbb{R}}}$, in $\mathbb{P}^2_{\mathbb{R}}$, such that $f^{-1}(V_-) \subset \tilde{V}_-$. Since $V_- \cup V_+$ is a neighborhood of $C \cap \mathbb{P}^2_{\mathbb{R}}$, we can find ϵ small enough such that $V_\epsilon \subset V_- \cup V_+$. From this we can deduce

$$f^{-1}(V_\epsilon \setminus V_+) \subset V_\epsilon. \tag{27}$$

Indeed, if X is in $V_\epsilon \setminus V_+$ then $X \in V_-$, and thus $f^{-1}(\{X\}) \subset \tilde{V}_-$. If Y is in $f^{-1}(\{X\})$, then by (26)

$$\epsilon \geq |r(\frac{X}{\|X\|})| = |r(\frac{R(Y)}{\|R(Y)\|})| \geq |r(\frac{Y}{\|Y\|})|.$$

Thus, Y is in V_ϵ.
This implies that if ϵ is small enough so that $(x_0, y_0) \notin V_\epsilon$, then the iterates $f^n((x_0, y_0))$ do not enter the set V_ϵ. Indeed, otherwise we can consider the first entrance time n_0 into V_ϵ. The point $f^{n_0}((x_0, y_0))$ is necessarily in $V_\epsilon \setminus V_+$ since (x_0, y_0) is not in the attractive basin of x_+. This implies that $f^{n_0-1}((x_0, y_0))$ is in V_ϵ, which is impossible.$\square$

Denote by $P^\infty_{\mathbb{R}}$ the line at infinity $P^\infty_{\mathbb{R}} = \{[x, y, 0], (x, y) \in \mathbb{R}^2\}$. Remark that the restriction of f to $P^\infty_{\mathbb{R}}$ is given by

$$f([x, y, 0]) = [x, \delta y, 0].$$

and that $P_{\mathbb{R}}^{\infty}$ is backward invariant, and $f^{-1}([x, y, 0]) = [x, \delta^{-1}y, 0]$. This implies that there exists an integer N_0 such that for any X in $\overline{P_{\mathbb{R}}^{\infty} \setminus V_\epsilon}$ the point $f^{-N_0}(X)$ is in V_ϵ. (This comes from the fact that V_ϵ contains a neighborhood of x_+ and x_-). Thus, we can find a neighborhood V_∞ of $\overline{P_{\mathbb{R}}^{\infty} \setminus V_\epsilon}$ such that $f^{-N_0}(V_\infty) \subset V_\epsilon$. We may as well take V_∞ such that $f^n((x_0, y_0)) \notin V_\infty$ when $n \leq N_0$. Let us now prove that the iterates $f^n((x_0, y_0))$ cannot enter the set V_∞: indeed, if $f^n((x_0, y_0)) \in V_\infty$ then $n \geq N_0$ and thus f^{n-N_0} is in V_ϵ which is in contradiction with the result we proved above. Hence, we proved that the iterates $f^n((x_0, y_0))$ do not enter neither V_ϵ nor V_∞. This concludes the proof of the lemma since $V_\epsilon \cup V_\infty$ contains a neighborhood of $P_{\mathbb{R}}^{\infty}$.□

We consider the operator $\tilde{\Gamma}_{<n>,\lambda}$ in $Sl_2(\mathbb{R})$ defined by

$$\tilde{\Gamma}_{<n>,\lambda} = (\delta)^{\frac{1}{2}}(D_\delta)^{-1}(D_{\sqrt{\delta}})^{-n}\Gamma_{<n>,\lambda}D^n_{\sqrt{\delta}},$$

and we set

$$\Pi_{<n>}(\lambda) = \prod_{k=0}^{n-1}(\sqrt{\delta}a(\gamma^k\lambda) + \sqrt{\delta}^{-1}d(\gamma^k\lambda)).$$

From (3) and (5) we have

$$\begin{aligned}
\Gamma_{<n>,\lambda} = \begin{pmatrix} a_{<n>}(\lambda) & b_{<n>}(\lambda) \\ c_{<n>}(\lambda) & d_{<n>}(\lambda) \end{pmatrix} &= \begin{pmatrix} a_{<n>}(\lambda) & \alpha^{-n}b(\gamma^n\lambda) \\ \alpha^n c(\gamma^n\lambda) & d_{<n>}(\lambda) \end{pmatrix} \\
&= \begin{pmatrix} a_{<n>}(\lambda) & \sqrt{\delta}^{-n}\Pi_{<n>}(\lambda)b(\lambda) \\ \sqrt{\delta}^n\Pi_{<n>}(\lambda)c(\lambda) & d_{<n>}(\lambda) \end{pmatrix}
\end{aligned}$$

thus, we get

$$\tilde{\Gamma}_{<n>,\lambda} = \begin{pmatrix} \sqrt{\delta}a_{<n>}(\lambda) & \sqrt{\delta}b(\lambda)\Pi_{<n>}(\lambda) \\ \sqrt{\delta}^{-1}c(\lambda)\Pi_{<n>}(\lambda) & \sqrt{\delta}^{-1}d_{<n>}(\lambda) \end{pmatrix}. \tag{28}$$

Since $\tilde{\Gamma}_{<n>,\lambda}$ is in $Sl_2(\mathbb{R})$, we deduce from lemma 4.2 that

$$|\Pi_{<n>}(\lambda)| \leq C_2 \tag{29}$$

where $C_2 = \frac{\sqrt{1+C_1^2}}{\sqrt{|b(\lambda)c(\lambda)|}}$ is a constant depending only on δ, λ (we know that $b(\lambda)$ and $c(\lambda)$ cannot be null, from proposition 2.1, since otherwise $\phi(\lambda)$ would be in the attractive bassin of x_+). We also deduce from lemma 4.2, formula (28) and relation (29) that there exists a constant C_3 such that for any n

$$C_3^{-1}\|X\|^2 \leq \|\tilde{\Gamma}_{<n>,\lambda}X\|^2 \leq C_3\|X\|^2, \tag{30}$$

for any vector X in $\mathbb{R}^2$ (and where $\|\ \|$ is the usual norm in $\mathbb{R}^2$).

Lemma 4.4 *For any λ in suppμ there exists a constant $C_4 > 0$ and a subsequence n_k such that for any solution f of (E_λ) on $I_{<n_k+1>}$ we have*

$$C_4^{-1} \leq \frac{\int_{I_{<n_k>}} |f|^2 dm_{n_k}}{\int_{I_{<n_k+1>}\setminus I_{<n_k>}} |f|^2 dm_{<n_k+1>}} \leq C_4. \tag{31}$$

Proof: Remark that it is enough to prove this result for $\omega = (1, \ldots, 1, \ldots)$ since the measure $m_{<n>}(\omega')$ is just, up to a scalar factor, the image of $m_{<n>}(\omega)$ by the right composition of Ψ_i and Ψ_i^{-1} that send the interval $I_{<n>}(\omega)$ to $I_{<n>}(\omega')$. Hence, we take $\omega = (1, \ldots, 1, \ldots)$ in this lemma.

We first introduce the bilinear form $K_{<n>,\lambda} : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ defined as follows: for (X, Y) in $\mathbb{R}^2 \times \mathbb{R}^2$ we denote by f and g the solutions of (E_λ) with initial conditions

$$\begin{pmatrix} f(0) \\ f'(0) \end{pmatrix} = X, \quad \begin{pmatrix} g(0) \\ g'(0) \end{pmatrix} = Y$$

and we set

$$K_{<n>,\lambda}(X, Y) = \int_{I_{<n>}} fg dm_{<n>}.$$

Clearly $K_{<n>,\lambda}$ is a positive definite symmetric bilinear form. We simply write $K_{<n>,\lambda}(X)$ for $K_{<n>,\lambda}(X, X)$ (it is possible to give an explicit expression of $K_{<n>,\lambda}$ in terms of the propagator and its derivative but we will not need it).

Take a solution f of (E_λ) on $I_{<n+1>}$. We denote by $\tilde{f}$ the function on $I_{<n>}$ equal to the "pull-back" of $f_{|I_{<n+1>} \setminus I_{<n>}}$ to $I_{<n>}$, i.e.

$$\tilde{f} = f_{|I_{<n+1>} \setminus I_{<n>}} \circ \Psi_1^{-(n+1)} \circ \Psi_2 \circ \Psi_1^n.$$

Clearly

$$\begin{pmatrix} \tilde{f}(0) \\ \tilde{f}'(0) \end{pmatrix} = (D_\delta)^{-1} \Gamma_{<n>,\lambda} \begin{pmatrix} f(0) \\ f'(0) \end{pmatrix},$$

and

$$\int_{I_{<n+1>}} |f|^2 dm_{<n+1>} = \int_{I_{<n>}} |f|^2 dm_{<n>} + \delta^{-1} \int_{I_{<n>}} |\tilde{f}|^2 dm_{<n>}.$$

This can be translated in

$$K_{<n+1>,\lambda}(X) = K_{<n>,\lambda}(X) + \delta^{-1} K_{<n>,\lambda}(D_\delta^{-1} \Gamma_{<n>,\lambda} X).$$

Define now

$$\tilde{K}_{<n>,\lambda}(X) = K_{<n>,\lambda}(D_{\sqrt{\delta}}^n X).$$

We see that the previous relation translates in

$$\tilde{K}_{<n+1>,\lambda}(X) = \tilde{K}_{<n>,\lambda}(D_{\sqrt{\delta}} X) + \tilde{K}_{<n>,\lambda}(\tilde{\Gamma}_{<n>,\lambda}(D_{\sqrt{\delta}} X)). \tag{32}$$

Take λ in suppμ, we first prove that there exists a constant C_5 and a sequence n_k such that

$$\frac{\sup_{\|X\|=1} \tilde{K}_{<n_k>,\lambda}(X)}{\inf_{\|X\|=1} \tilde{K}_{<n_k>,\lambda}(X)} \leq C_5. \tag{33}$$

This comes from the following technical lemma

Lemma 4.5 *If K is a positive quadratic form on $\mathbb{R}^2$ and Γ an element of $Sl_2(\mathbb{R})$ with $|Tr\Gamma| < 2$ then we have the inequality*

$$(\sup_{\|X\|=1} K(X)) \left(\frac{1 - |Tr\Gamma|^2/4}{\|\Gamma\|^2} \right)^2 \leq K(Z) + K(\Gamma Z) \leq (\sup_{\|X\|=1} K(X))(1 + \|\Gamma\|^2).$$

for all Z in $\mathbb{R}^2$ with $\|Z\| = 1$. (In the last formula $\|\Gamma\|^2 = Tr\Gamma^*\Gamma$).

Proof: The inequality of the right hand side is trivial. For the left hand side inequality we may as well suppose that K is diagonal, i.e. that K is of the form $K(X) = \rho_1 x^2 + \rho_2 y^2$ for $X = \begin{pmatrix} x \\ y \end{pmatrix}$ (indeed, by an orthogonal change of variables O, the operator $O^*\Gamma O$ remain in $\mathrm{Sl}_2(\mathbb{R})$ and keeps the same trace). We suppose that $\rho_1 \geq \rho_2$. If

$$\Gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

and $Z = \begin{pmatrix} x \\ y \end{pmatrix}$, with $x^2 + y^2 = 1$, then

$$\begin{aligned} K(Z) + K(\Gamma Z) &\geq \rho_1(x^2 + (ax+by)^2) \\ &\geq \rho_1 \frac{1}{\| \begin{pmatrix} 1 & 0 \\ a & b \end{pmatrix}^{-1} \|^2} \\ &\geq \rho_1 \frac{b^2}{\|\Gamma\|^2}. \end{aligned}$$

Suppose now that $ad \leq 0$ then $|bc| \geq 1$ and thus $|b|^2 \geq \frac{1}{\|\Gamma\|^2}$.
Suppose now that $ad \geq 0$ then $ad \leq \mathrm{Tr}(\Gamma)^2/4$ thus $|bc| \geq (1 - \mathrm{Tr}(\Gamma)^2/4)$ and we get the left hand side inequality.□

We apply this result to $K_{<n>,\lambda}$. Remark first that $\mathrm{Tr}(\Gamma_{<n>,\lambda}) = \sqrt{\delta} a_{<n>}(\lambda) + \sqrt{\delta}^{-1} d_{<n>}(\lambda)$. From the definition of $\Pi_{<n>}(\lambda)$ and relation (29), we deduce that there exists a subsequence n_k such that $|\mathrm{Tr}\tilde{\Gamma}_{<n_k-1>,\lambda}| \leq \frac{2}{\sqrt{3}}$, for example. From relation (32) and the previous lemma we easily deduce relation (33) with $C_5 = \frac{3}{2}\delta C_3^2(1 + C_3)$.

We are now in position to conclude the proof of lemma 4.4. Consider a function f, solution of (E_λ) on $I_{<n_k+1>}$. Set

$$X = D_{\sqrt{\delta}}^{-n} \begin{pmatrix} f(0) \\ f'(0) \end{pmatrix}.$$

By definition we have

$$\int_{I_{<n_k>}} |f|^2 dm_{<n_k>} = \tilde{K}_{<n_k>,\lambda}(X), \quad \text{and}$$

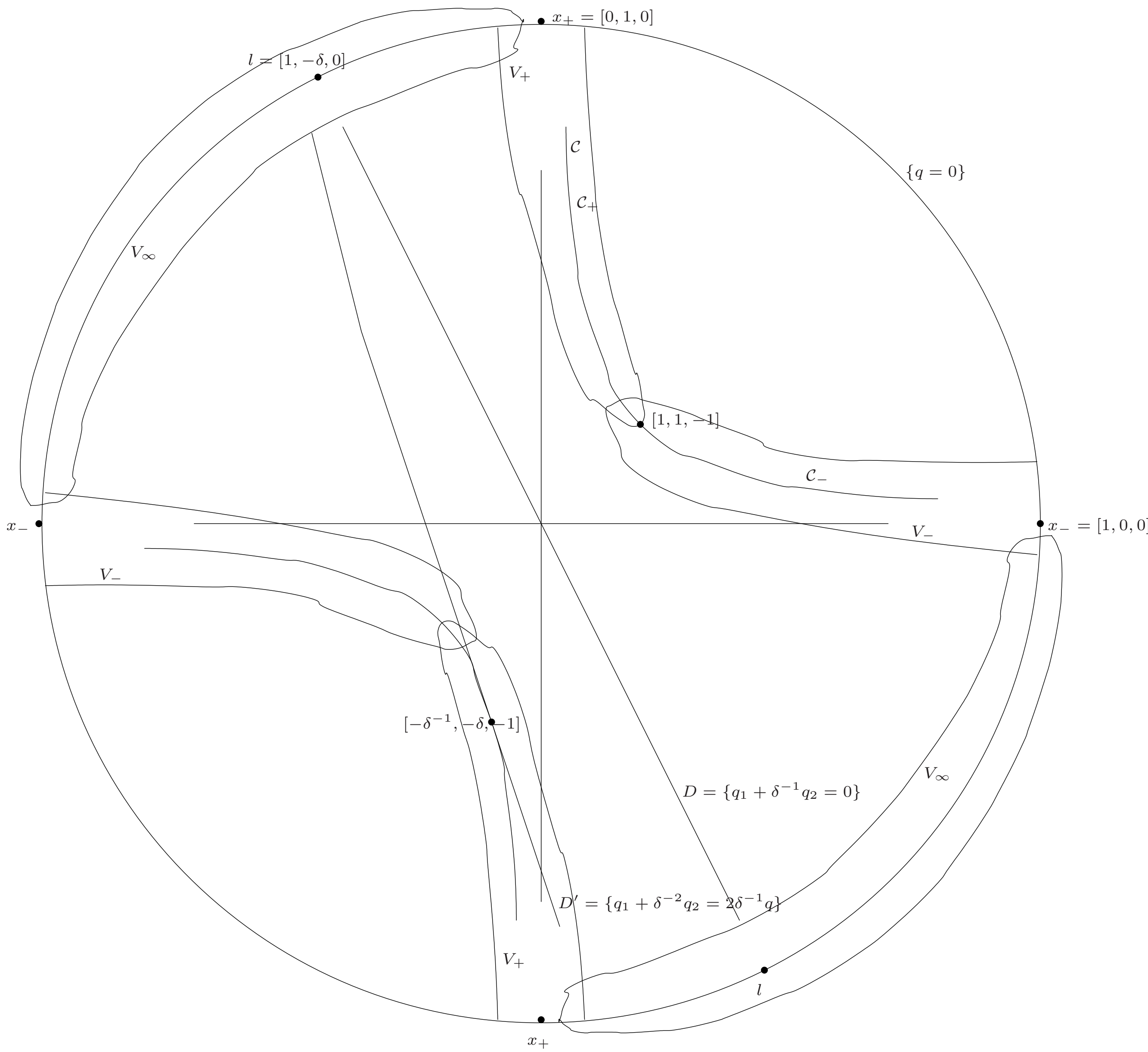
$$\int_{I_{<n_k+1>} \setminus I_{<n_k>}} |f|^2 dm_{<n_k+1>} = \tilde{K}_{<n_k>,\lambda}(\tilde{\Gamma}_{<n_k>,\lambda} X).$$

From relation (33) and relation (30) we see that the inequality of lemma 4.4 is satisfied for $C_4 = C_3^2 C_5$.□

We can now finish the proof of lemma 4.1. Take any blow-up ω. Suppose that λ in $\mathrm{supp}\mu$ is an eigenvalue of $H^{\pm}_{<\infty>}(\omega)$ with eigenfunction f, $\|f\| = 1$. For any $\epsilon > 0$ there exists N such that

$$\int_{I_{<\infty>} \setminus I_{<N>}} |f|^2 dm_{<\infty>} \leq \epsilon.$$

Clearly, this contradicts lemma 4.4 for ϵ small enough. This concludes the proof of theorem 3.1.◇



Picture of $\mathbb{P}^2_{\mathbb{R}}$ for $\delta = 2$

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